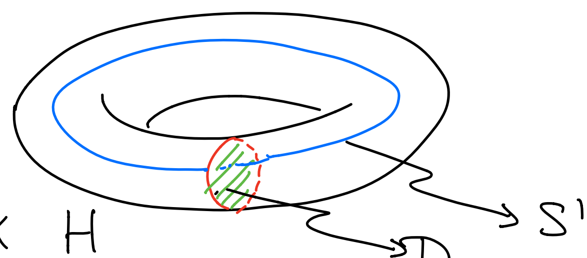
§10. Witten's invariants for 3-manifolds

Let L be a framed link in S^3.

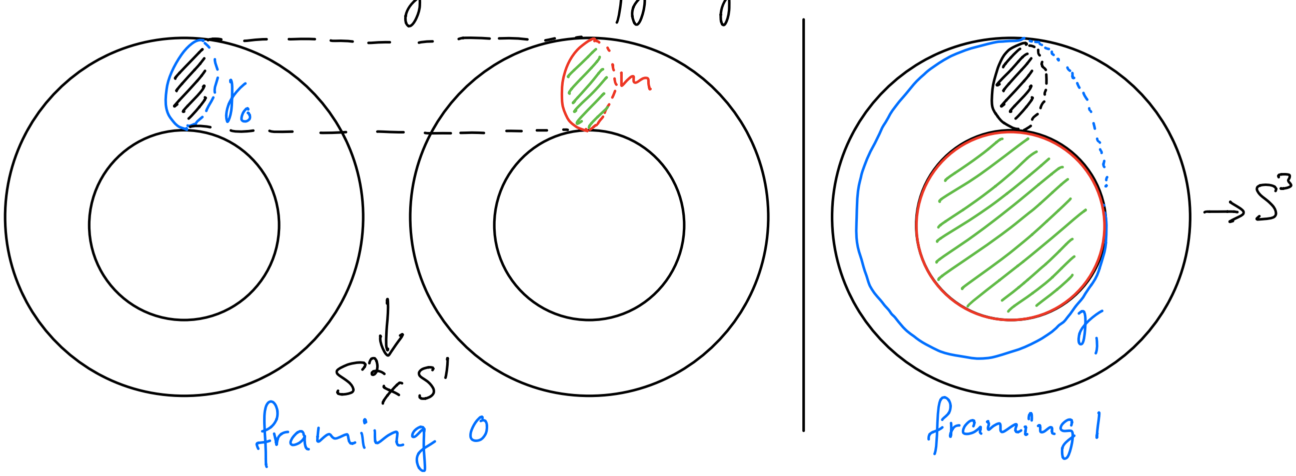
Dehn surgery:

Take for simplicity L to be the unknot. Consider tubular neighborhood $N(L)$ of L. $N(L)$ is homeomorphic to $D \times S^1$. Take closed curve γ on $\partial N(L)$ giving the framing of L. Let m be the meridian on the boundary of $H \cong D \times S^1$.

Put $E(L) = \overline{S^3 \setminus N(L)}$



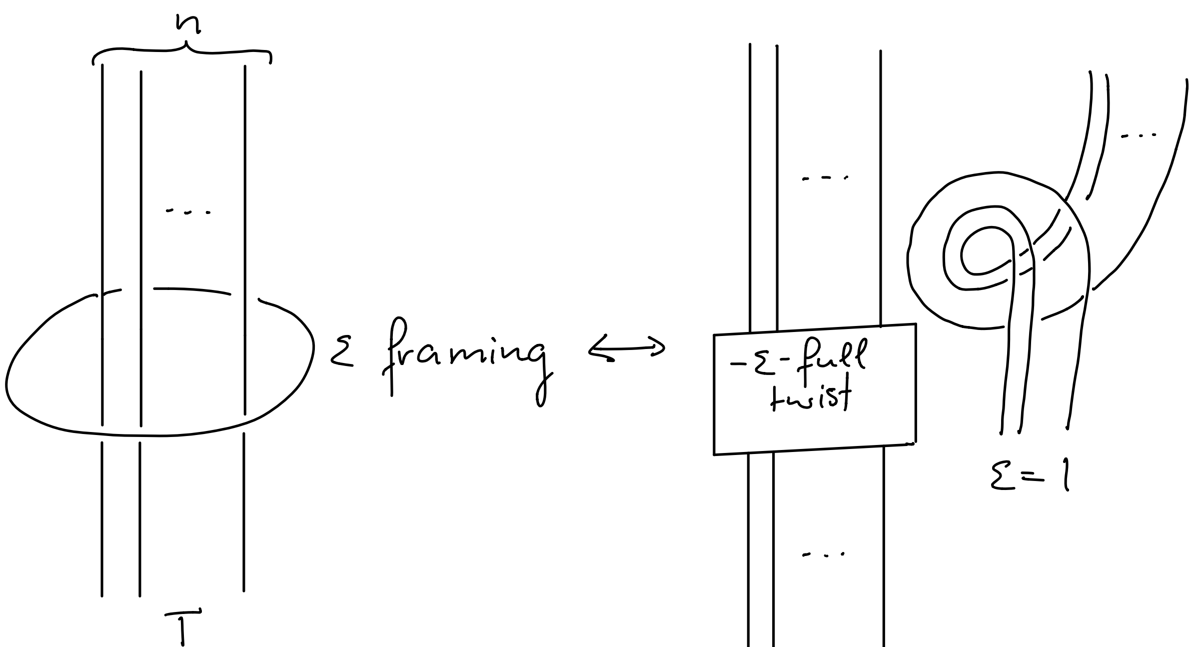
$E(L)$ is a solid torus itself! We glue back H into $E(L)$ by identifying m with γ



Let L and L' be framed links in S^3. Denote by M_L and M_L' the 3-manifolds obtained by Dehn surgery on L and L', respectively. Then we have the following

<u>Theorem 1</u> (Kirby moves):

There is an orientation preserving homeom. $M_L \cong M_L'$ if and only if L' is obtained from L by applying the following local move



finitely many times, where $\varepsilon = \pm 1$ and n stand for the number of strands passing

through the trivial knot with ε-framing.
For $\varepsilon = 0 \Rightarrow$ deleting / adding trivial knot with ε framing

Next: Define Witten's invariants for arbitrary 3-manifolds obtained by Dehn surgery from S^3.

Let $\mathfrak{g}$ be the Lie algebra $sl_2(\mathbb{C})$. Fix a positive integer K and denote by $P_+(K)$ the set of level K highest weights of affine Lie algebra $\hat{\mathfrak{g}}$.

$\longrightarrow P_+(K) = \{0, 1, \dots, K\}$. For each $\lambda \in P_+(K) \to H_\lambda$ on which Virasoro algebra acts with central charge $c = \frac{3K}{K+2}$. Set $C = \exp\left(2\pi\sqrt{-1}\,\frac{c}{24}\right)^{-3} = e^{-\pi\sqrt{-1}\frac{c}{4}}$

Level K characters $\chi_\lambda(\tau)$, $\operatorname{Im}\tau > 0$, $\lambda \in P_+(K)$ satisfy:

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \sum_\mu S_{\lambda\mu}\chi_\mu(\tau),$$

$$\chi_\lambda(\tau+1) = \exp\left(2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)\right)\chi_\lambda(\tau),$$

where $S_{\lambda\mu} = \sqrt{\frac{2}{K+2}}\,\sin\frac{(\lambda+1)(\mu+1)}{K+2}$, $\Delta_\lambda = \frac{\lambda(\lambda+2)}{4(K+2)}$

Modular transformations S and T satisfy: $S^2 = (ST)^3 = \mathbb{1}$

As a consequence of the above we have

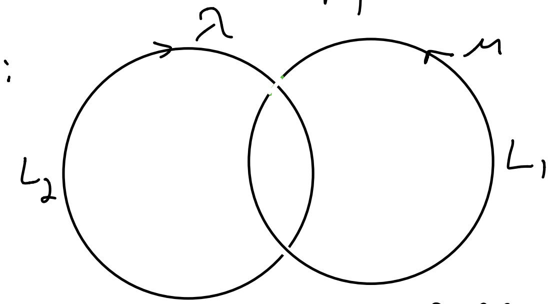
<u>Lemma 1:</u>

The above $S_{\lambda\mu}$, $0 \le \lambda, \mu \le k$, satisfy

$$C \sum_{\mu} S_{\lambda\mu} S_{\mu\nu} \exp\left(2\pi\sqrt{-1}\left(\Delta_\lambda + \Delta_\mu + \Delta_\nu\right)\right) = S_{\lambda\nu}$$

Let now L be an oriented framed link in S^3 with components $L_1, \dots, L_m$. Given a coloring $\lambda : \{1, \dots, n\} \longrightarrow P_+(k)$ with highest weights of level $k \longrightarrow$ invariants $\mathcal{J}(L; \lambda_1, \dots, \lambda_m)$

Let L be the Hopf link with two components L_1, L_2:



$lk(L_1, L_2) = 1$

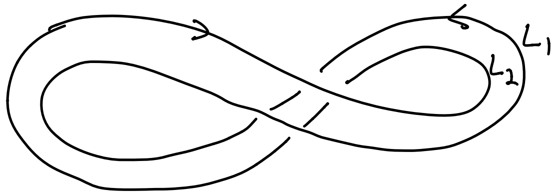
Then we have the following

<u>Proposition 1:</u>

Let H be a Hopf link colored with $\lambda, \mu \in P_+(k)$. Then $\mathcal{J}(H; \lambda, \mu) = \dfrac{S_{\lambda\mu}}{S_{00}}$

Proof:

Represent the Hopf link as a cabling of a trivial knot with -1 framing:



$f(L_1) = -1$

$f(L_2) = -1$

$lk(L_1, L_2) = -1$

By Lemma 3 and Proposition 1 of §9, we have

$$\exp 2\pi\sqrt{-1}\,(-\Delta_\lambda - \Delta_\mu)\, J(H; \lambda, \mu)$$

$$= \sum_\nu N_{\lambda\mu}^{\nu} \exp\left(2\pi\sqrt{-1}\,(-\Delta_\nu)\right) \frac{S_{0\nu}}{S_{00}}$$

where we have used that $J(\bigcirc; \nu) = \dfrac{S_{0\nu}}{S_{00}}$.

Then, using Verlinde's formula and Lemma 1, we compute:

$$N_{\lambda\mu\nu} = \sum_\alpha \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \quad \text{(Verlinde formula)}$$

$$\Longrightarrow \sum_\nu N_{\lambda\mu}^{\nu} \exp\left(2\pi\sqrt{-1}\,(-\Delta_\nu)\right) \frac{S_{0\nu}}{S_{00}}$$

$$= \sum_{\nu,\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \frac{S_{0\nu}}{S_{00}} \exp\left(2\pi\sqrt{-1}\,(-\Delta_\nu)\right)$$

$$\Rightarrow e^{2\pi i(-\Delta_\lambda - \Delta_\mu)} J(H; \lambda, \mu)$$

$$= \sum_{\nu, \alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha} S_{0\nu}}{S_{0\alpha} S_{00}} e^{-2\pi i \Delta_\nu}$$

$$\Leftrightarrow J(H; \lambda, \mu) = \sum_{\nu, \alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha} S_{0\nu}}{S_{0\alpha} S_{00}} e^{-2\pi i(\Delta_\nu - \Delta_\lambda - \Delta_\mu)}$$

$$\sum_\nu S_{0\nu} S_{\nu\alpha} e^{-2\pi i \Delta_\nu} = C S_{0\alpha} e^{2\pi i \Delta_\alpha}$$

(rearranging Lemma 1)

$$= C \sum_\alpha \frac{S_{\lambda\alpha} S_{\mu\alpha} \cancel{S_{0\alpha}}}{\cancel{S_{0\alpha}} S_{00}} e^{2\pi i(\Delta_\alpha + \Delta_\lambda + \Delta_\mu)}$$

Lemma 1 again

$$= \frac{S_{\lambda\mu}}{S_{00}}$$

$\square$

<u>Notation:</u>

For link components L_i and L_j we write $L_i \cdot L_j = lk(L_i, L_j)$. In the case $i=j$, $L_i \cdot L_i$ denotes the integer representing the framing of L_i $\longrightarrow$ obtain matrix A $(A_{ij} = (L_i \cdot L_j))$

Let n_+ (resp. n_-) be the number of positive (resp. neg.) eigenvalues of A. Then we write

$$\sigma(L) = n_+ - n_-$$

for the signature of the link L.

Theorem 2:

Let M be a compact oriented 3-manifold without boundary. Suppose that M is obtained as Dehn surgery on a framed link L with m components L_j, $1 \leq j \leq m$, in S^3. Then,

$$Z_K(M) = S_{00}\, C^{\sigma(L)} \sum_{\{\lambda\}} S_{0\lambda_1} \cdots S_{0\lambda_m} J(L; \lambda_1, \dots, \lambda_m)$$

is a topological invariant of M and does not depend on the choice of L which yields M. More precisely, if there is an orientation preserving homeomorphism $M_1 \cong M_2$, then $Z_K(M_1) = Z_K(M_2)$. $Z_K(M)$ is the Chern-Simons partition function of M.